The Einstein-Maxwell Equations, Extremal Kähler Metrics, and Seiberg-Witten Theory

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Nigel Hitchin has played a key role in the exploration of 4-dimensional Riemannian geometry, and in particular (Atiyah, Hitchin & Singer 1978, Hitchin 1974*a*, 1974*b*, 1975, 1979, 1981, Hitchin, Karlhede, Lindström & Roček 1987) has made foundational contributions to the theory of self-dual manifolds, 4-dimensional Einstein manifolds, spin^c structures, and Kähler geometry. In the process, he has often alerted the rest of us to the profound mathematical interest of beautiful geometric problems that had previously only been considered by physicists. I would therefore like to use the occasion of Nigel's 60$^{\text{th}}$ birthday as an opportunity to draw the attention of an audience of geometers and physicists to an interesting relationship between the 4-dimensional Einstein-Maxwell equations and Kähler geometry, and point out some fascinating open problems that directly impinge on this relationship.

Let us begin by recalling that a 2-form F on an oriented pseudo-Riemannian n-manifold (M, g) is said to satisfy *Maxwell's equations* (in vacuo) if and only if

$$\begin{aligned} dF &= 0 \\ d\star F &= 0 \end{aligned}$$

where $\star$ is the Hodge star operator. If M is compact and g is Riemannian, these equations of course just mean that F is a harmonic 2-form, and Hodge theory thus asserts that there is in fact exactly one solution in each de Rham

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cohomology class $[F] \in H^2(M, \mathbb{R})$. This solution may be found by minimizing the action

$$F \longmapsto \int_M |F|_g^2 d\mu_g$$

among all closed forms $F \in [F]$. In this context, dimension four enjoys a somewhat privileged status, because it is precisely when $n = 4$ that both the action and the solutions themselves become conformally invariant, in the sense that they are unaltered by replacing g with any conformally related metric $\tilde{g} = u^2 g$.

When $n > 2$, coupling these equations to the gravitational field (Hawking & Ellis 1973, Misner, Thorne & Wheeler 1973) gives rise to the so-called *Einstein-Maxwell equations* (with cosmological constant)

$$\begin{aligned} dF &= 0 \\ d \star F &= 0 \\ \Big[r + F \circ F \Big]_0 &= 0 \end{aligned}$$

where r is the Ricci tensor of g, where $(F \circ F)_{jk} = F_j{}^\ell F_{\ell k}$ is obtained by composing F with itself as an endomorphism of TM, and where $[\ ]_0$ indicates the trace-free part with respect to g. In the compact Riemannian setting, these equations may be understood as the Euler-Lagrange equations of the functional

$$(g, F) \longmapsto \int_M (s_g + |F|_g^2) d\mu_g$$

where F is again allowed to vary over all closed 2-forms in a given de Rham class, and where g is allowed to vary over all Riemannian metrics of some fixed total volume V. The privileged status of dimension four becomes more pronounced in this context, for it is only when $n = 4$ that the Einstein-Maxwell equations imply that the scalar curvature s of g is *constant*. Indeed, this just reflects Yamabe's observation (Yamabe 1960) that a Riemannian metric has constant scalar curvature iff it is a critical point of the restriction of the Einstein-Hilbert action $\int s \, d\mu$ to the space of volume-V metrics in a fixed conformal class. When $n = 4$, the conformal invariance of $\int |F|^2 d\mu$ thus implies that critical points of the above functional must have constant scalar curvature; but when $n \neq 4$, by contrast, the scalar curvature turns out to be constant only when F has constant norm.

We have just observed that the Einstein-Maxwell equations on a 4-manifold imply that the scalar curvature is constant. But what happens in

the converse direction is far more remarkable: namely, *any constant-scalar-curvature Kähler metric on a* 4*-manifold may be interpreted as a solution of the Einstein-Maxwell equations.* Indeed, suppose that (M^4, g, J) is a Kähler surface, with Kähler form $\omega = g(J\cdot, \cdot)$ and Ricci form $\rho = r(J\cdot, \cdot)$. Let

$$\mathring{\rho} = \rho - \frac{s}{4}\omega$$

denote the primitive part of the Ricci form, corresponding to the trace-free Ricci tensor

$$\mathring{r} := [r]_0 = r - \frac{s}{4}g.$$

Suppose that the scalar curvature s of (M, g) is constant, and set

$$F = \omega + \frac{\mathring{\rho}}{2}.$$

Then (g, F) automatically solves the Einstein-Maxwell equations. This generalizes an observation due to Flaherty (1978) concerning the scalar-flat case.

On a purely calculational level, this fact is certainly easy enough to check. Indeed, on any oriented Riemannian 4-manifold, the 2-forms canonically decompose

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

into self-dual and anti-self-dual parts, and it is then easily shown that for any 2-form

$$F = F^+ + F^-$$

one has

$$\Big[F \circ F\Big]_0 = 2F^+ \circ F^-.$$

Since in our special case we have $F^+ = \omega$ and $F^- = \mathring{\rho}/2$, it therefore follows that

$$\Big[F \circ F\Big]_0 = -\mathring{r},$$

so that

$$\Big[r + F \circ F\Big]_0 = 0$$

as required. Moreover, since ρ is automatically closed, and its self-dual part $s\omega/4$ is closed if s is assumed to be constant, we conclude that F is indeed harmonic on a constant-scalar-curvature Kähler surface, exactly as claimed.

But there is actually a great deal more going on here. Recall that Calabi (1982) defined an *extremal Kähler metric* on a compact complex manifold (M^{2m}, J) to be a Kähler metric which is a critical point of the Riemannian functional

$$g \longmapsto \int_M s^2 \, d\mu$$

restricted to a fixed Kähler class $[\omega] \in H^2(M)$. The associated Euler-Lagrange equations then amount to requiring that the gradient ∇s of the scalar curvature be the real part of a holomorphic vector field. In particular, any constant-scalar-curvature Kähler metric is extremal in this sense. In fact, as was recently proved by Chen (2006), extremal Kähler metrics actually always *minimize* $\int s^2 d\mu$ in their Kähler classes. In the constant-scalar-curvature case, this was long ago shown by Calabi, using a simple but elegant argument. Indeed, if (M^{2m}, g, J) is a compact Kähler manifold of complex dimension m, then

$$\int_M s_g \, d\mu_g = \int_M \rho \wedge \star\omega = \frac{4\pi}{(m-1)!} c_1 \cdot [\omega]^{m-1}$$

and

$$\int_M 1 \, d\mu = \int_M \frac{\omega^{\wedge m}}{m!} = \frac{1}{m!}[\omega]^m$$

so that the Cauchy-Schwarz inequality tells us that

$$\int_M s^2 d\mu \geq \frac{(\int s \, d\mu)^2}{\int 1 \, d\mu} = \frac{16\pi^2 m}{(m-1)!} \frac{(c_1 \cdot [\omega]^{m-1})^2}{[\omega]^m} \tag{1}$$

with equality iff s is constant.

Calabi (1982) also considered the Riemannian functionals

$$g \longmapsto \int_M |r|_g^2 \, d\mu_g$$

and

$$g \longmapsto \int_M |\mathcal{R}|_g^2 \, d\mu_g$$

obtained by squaring the L^2-norms of the Ricci curvature r and the full Riemann curvature $\mathcal{R}$. Here, his observation was that the restriction of

either of these functionals to the space of Kähler metrics can be rewritten in the form

$$g \longmapsto a + b \int_M s^2 \, d\mu$$

where a and $b > 0$ are constants depending only on the Kähler class. For example,

$$\int_M |r|_g^2 d\mu_g = \frac{1}{2} \int_M s_g^2 \, d\mu_g - \frac{8\pi^2}{(m-2)!} c_1^2 \cdot [\omega]^{m-2},$$

so that

$$\int_M |r|_g^2 d\mu_g \geq \frac{8\pi^2}{(m-2)!} \left[\frac{m}{m-1} \frac{(c_1 \cdot [\omega]^{m-1})^2}{[\omega]^m} - c_1^2 \cdot [\omega]^{m-2} \right], \qquad (2)$$

with equality iff s is constant. Thus extremal Kähler metrics turn out to minimize these functionals, too.

I would now like to point out an interesting Riemannian analog of Calabi's variational problem that leads to the Einstein-Maxwell equations on a smooth compact 4-manifold. To this end, first notice that the Kähler form of a Kähler surface is self-dual and harmonic. Let us therefore introduce the following notion:

Definition 1 *Let M be smooth compact oriented 4-manifold, and let $[\omega] \in H^2(M, \mathbb{R})$ be a deRham class with $[\omega]^2 > 0$. We will then say that a Riemannian metric g is* adapted *to $[\omega]$ if the harmonic form ω representing $[\omega]$ with respect to g is* self-dual.

This allows us to introduce the Riemannian analog of a Kähler class:

Definition 2 *Let M be smooth compact oriented 4-manifold, and let $[\omega] \in H^2(M, \mathbb{R})$ be a deRham class with $[\omega]^2 > 0$. We then set*

$$\mathcal{G}_{[\omega]} = \Big\{ \textit{smooth metrics } g \textit{ on } M \textit{ which are adapted to } [\omega] \Big\}.$$

In particular, if ω is the Kähler form of a metric g on M which is Kähler with respect to some complex structure J, then $\mathcal{G}_{[\omega]}$ contains the entire Kähler class of ω on (M, J). Of course, however, $\mathcal{G}_{[\omega]}$ is vastly larger than a Kähler class. In particular, if g belongs to $\mathcal{G}_{[\omega]}$, so does every conformally related metric $\tilde{g} = u^2 g$. It is also worth noticing that

$$\mathcal{G}_{\lambda[\omega]} = \mathcal{G}_{[\omega]}$$

for any non-zero real constant λ.

It is now germane to ask precisely how large $\mathcal{G}_{[\omega]}$ really is relative to

$$\mathcal{G} = \Big\{\text{smooth metrics } g \text{ on } M\Big\},$$

and to ponder the dependence of $\mathcal{G}_{[\omega]} \subset \mathcal{G}$ on $[\omega]$ as we allow this cohomology class to vary through the open cone

$$\mathcal{C} = \Big\{[\omega] \in H^2(M,\mathbb{R}) \;\Big|\; [\omega]^2 > 0\Big\}.$$

Proposition 1 (Donaldson/Gay-Kirby) *Let M be any smooth compact 4-manifold with $b_+(M) \neq 0$. For any $[\omega] \in \mathcal{C}$, $\mathcal{G}_{[\omega]} \subset \mathcal{G}$ is a Fréchet submanifold of finite codimension $b_-(M)$. Moreover, $\mathcal{G}_{[\omega]} \neq \emptyset$ for all $[\omega]$ belonging to an open dense subset of $\mathcal{C}$.*

Indeed, if $g \in \mathcal{G}_{[\omega]}$ and if $\omega \in [\omega]$ is the harmonic representative, then Donaldson (1986, p. 336) has shown that $T_g\mathcal{G}_{[\omega]}$ is precisely the L^2-orthogonal of the $b_-(M)$-dimensional subspace

$$\{\omega \circ \varphi \mid \varphi \in \mathcal{H}^-_g\} \subset \Gamma(\odot^2 T^*M),$$

where $\mathcal{H}^-_g$ is the space of anti-self-dual harmonic 2-forms with respect to g; moreover, his proof also shows that the subset of $[\omega] \in \mathcal{C}$ for which $\mathcal{G}_{[\omega]} \neq \emptyset$ is necessarily open. On the other hand, Gay and Kirby (2004) found an essentially explicit way of constructing a metric g adapted to any $[\omega] \in \mathcal{C} \cap H^2(M,\mathbb{Z})$, so that $\mathcal{G}_{[\omega]} \neq \emptyset$ for any $[\omega]$ in the dense subset $[\mathcal{C} \cap H^2(M,\mathbb{Q})] \subset \mathcal{C}$.

We now consider the natural generalization of Calabi's variational problem to this broader context.

Proposition 2 *An $[\omega]$-adapted metric g is a critical point of the Riemannian functional*

$$g \longmapsto \int_M s_g^2 d\mu_g$$

restricted to $\mathcal{G}_{[\omega]}$ iff either

- *g is a solution of the Einstein-Maxwell equations, in conjunction with a unique harmonic form F with $F^+ = \omega$; or else*
- *g is scalar-flat ($s \equiv 0$).*

Proof. Consider a 1-parameter family of metrics

$$g_t := g + th + O(t^2)$$

in $\mathcal{G}_{[\omega]}$. By Donaldson's result, we know that h can be taken to be any smooth symmetric tensor field which satisfies

$$\int_M \langle h, \omega \circ \varphi \rangle d\mu = 0$$

for all harmonic forms $\varphi \in \Gamma(\Lambda^-)$, where ω is the g-harmonic representative of $[\omega]$. On the other hand, a standard calculation (Besse 1987) shows that

$$\left. \frac{d}{dt} s \right|_{t=0} = \Delta h_a^a + \nabla^a \nabla^b h_{ab} - h^{ab} r_{ab},$$

and

$$\left. \frac{d}{dt} [d\mu] \right|_{t=0} = \frac{1}{2} h_a^a d\mu,$$

so that

$$\begin{aligned} \left. \frac{d}{dt} \left[\int_M s^2 d\mu \right] \right|_{t=0} &= \int_M 2s\dot{s} d\mu + \int_M s^2 \dot{d\mu} \\ &= \int 2s \left(\Delta h_a^a + \nabla^a \nabla^b h_{ab} - h^{ab} \mathring{r}_{ab} \right) d\mu \end{aligned}$$

where $\mathring{r}$ again denotes the trace-free part of the Ricci tensor.

Let us now ask when a metric is critical within its conformal class. This corresponds to setting $h = vg$ for some smooth function v. We then have

$$\frac{d}{dt} \int_M s^2 d\mu = \int 2s(3\Delta v) d\mu = 6 \int_M \langle ds, dv \rangle d\mu,$$

so the derivative is zero for all such variations iff s is constant.

We may thus assume henceforth that s is constant. We then have

$$\begin{aligned} \frac{d}{dt} \int_M s^2 d\mu &= 2s \int \left(\Delta h_a^a + \nabla^a \nabla^b h_{ab} - h^{ab} \mathring{r}_{ab} \right) d\mu \\ &= -2s \int_M \langle h, \mathring{r} \rangle d\mu. \end{aligned}$$

If $s \equiv 0$, this obviously vanishes for every h, and g is a critical point. Otherwise, g will be critical iff $\mathring{r}$ belongs to the L^2-orthogonal complement of $T_g\mathcal{G}_{[\omega]}$. But we already have seen that this orthogonal complement precisely consists of tensors of the form $\omega \circ \varphi$, $\varphi \in \mathcal{H}_g^-$. Thus, when $s \not\equiv 0$, g is a critical point iff s is constant and $\mathring{r} = \omega \circ \varphi$ for some $\varphi \in \mathcal{H}_g^-$. But, setting

$$F = \omega + \frac{\varphi}{2} \ ,$$

this is in turn equivalent to saying that (g, F) solves the Einstein-Maxwell equations, as claimed. ∎

So, why are constant-scalar-curvature Kähler metrics critical points of $\int s^2 d\mu$ restricted to $\mathcal{G}_{[\omega]}$? Well, we will now see that they typically turn out not only to be critical points, but actually to be *minima*. Indeed, the following result (LeBrun 1995, LeBrun 2001, LeBrun 2003) may be thought of as a Riemannian generalization of Calabi's inequalities (1–2):

Theorem 1 *Let (M^4, J) be a compact complex surface, and suppose that $[\omega]$ is a Kähler class with $c_1 \cdot [\omega] \leq 0$. Then any* Riemannian *metric $g \in \mathcal{G}_{[\omega]}$ satisfies the inequalities*

$$\int s^2 d\mu \quad \geq \quad 32\pi^2 \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \tag{3}$$

$$\int |r|^2 d\mu \quad \geq \quad 8\pi^2 \Big[2\frac{(c_1 \cdot [\omega])^2}{[\omega]^2} - c_1^2 \Big] \tag{4}$$

with equality if and only if g is constant-scalar-curvature Kähler.

In the equality case, the complex structure $\tilde{J}$ with respect to which g is Kähler will typically be different from J, but must have the same first Chern class c_1, while its Kähler class must be a positive multiple of $[\omega]$.

We also remark that if (M, J) is not rational or ruled, the hypothesis that $c_1 \cdot [\omega] \leq 0$ holds automatically, and that in this setting a Kähler metric is extremal iff it has constant scalar curvature. In this context, the relevant constant is of course necessarily non-positive.

By contrast, if (M, J) is rational or ruled, there will always be Kähler classes for which $c_1 \cdot [\omega] > 0$. When this happens, the above generalization

(3) of (1) turns out definitely *not* to hold for arbitrary Riemannian metrics. Instead, the correct generalization (LeBrun 1997) is that

$$Y_{[g]} \leq \frac{4\pi\ c_1 \cdot [\omega]}{\sqrt{[\omega]^2/2}}\ , \tag{5}$$

where the Yamabe constant $Y_{[g]}$ is obtained by minimizing the Einstein-Hilbert action $\int s\ d\mu$ over all unit-volume metrics $\tilde{g} = u^2 g$ conformal to g. Moreover, the inequality is strict unless the Yamabe minimizer is a constant-scalar-curvature Kähler metric, so that (3) is in fact violated by an appropriate conformal rescaling of any generic Riemannian metric of positive scalar curvature.

It is also worth remarking that no sharp lower bound in the spirit of Theorem 1 is currently known for the square-norm $\int |\mathcal{R}|^2 d\mu$ of the Riemann curvature tensor. Deriving one would be extremely interesting and potentially very useful, but, for reasons I will now explain, the technical obstacles to doing so seem formidable.

Recall that, by raising an index, the Riemann curvature tensor may be reinterpreted as a linear map $\Lambda^2 \to \Lambda^2$, called the *curvature operator*. The decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ thus allows one to view this linear map as consisting of four blocks:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right) . \tag{6}$$

Here $W_\pm$ are the trace-free pieces of the appropriate blocks, and are called the self-dual and anti-self-dual Weyl curvatures, respectively. The scalar curvature s is understood to act by scalar multiplication, whereas the trace-free Ricci curvature $\mathring{r} = r - \frac{s}{4}g$ acts on 2-forms by $\varphi_{ab} \mapsto\ 2\varphi_{[a}{}^{c}\mathring{r}_{b]c}$.

When (M, g) happens to be Kähler, $\Lambda^{2,0} \subset \ker \mathcal{R}$, and the entire upper-left-hand block is therefore entirely determined by the scalar curvature s. For Kähler metrics, one thus obtains the identity

$$|W_+|^2 \equiv \frac{s^2}{24},$$

and Gauss-Bonnet-type formulæ like

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

reduce many questions about square-norms of curvature to questions about the scalar-curvature alone. But for general Riemannian metrics, the norms of s and W_+ are utterly independent quantities, so if one wants to use the identity

$$\int |r|^2 d\mu = -8\pi^2(2\chi + 3\tau)(M) + 8\int \left(\frac{s^2}{24} + \frac{1}{2}|W_+|^2\right) d\mu \tag{7}$$

to prove a generalization of (2) for Riemannian metrics, information must be obtained concerning not only the scalar curvature, but also concerning the self-dual Weyl curvature as well.

The curvature estimates of Theorem 1 are derived by means of Seiberg-Witten theory (Witten 1994), making it clear that this really is an essentially 4-dimensional story. The complex structure J determines a spin^c structure on M with twisted spin bundles $\mathbb{S}_\pm \otimes L^{1/2}$, where L^{-1} is the canonical line bundle $\Lambda^{2,0}$ of (M, J). For simplicity, suppose that $c_1 \cdot [\omega] < 0$. For each metric $g \in \mathcal{G}_{[\omega]}$, one then considers the Seiberg-Witten equations

$$\begin{aligned} D_A\Phi &= 0 \\ F_A^+ &= -\frac{1}{2}\Phi \odot \bar{\Phi} \end{aligned}$$

where the unknowns are a unitary connection A on the line-bundle $L \to M$ and a twisted spinor $\Phi \in \Gamma(\mathbb{S}_+ \otimes L^{1/2})$; here $D_A : \Gamma(\mathbb{S}_+ \otimes L^{1/2}) \to \Gamma(\mathbb{S}_- \otimes L^{1/2})$ denotes the twisted Dirac operator associated with A, and F_A^+ is the self-dual part of the curvature of A. One then shows that there must be at least one solution for each $g \in \mathcal{G}_{[\omega]}$ by establishing a count of solutions modulo gauge equivalence which is independent of the metric and which is obviously non-zero for a Kähler metric.

However, the Seiberg-Witten equations can be shown to imply various curvature estimates via Weitzenböck formulæ. In particular, the existence of at least one solution for each metric $\tilde{g} = u^2 g$ conformal to g is enough to guarantee that the curvature of g satisfies

$$\begin{aligned} \int_M s^2 d\mu_g &\geq 32\pi^2[c_1(L)^+]^2 \\ \int_M \left(s - \sqrt{6}|W_+|\right)^2 d\mu_g &\geq 72\pi^2[c_1(L)^+]^2 \end{aligned}$$

where $[c_1(L)]^+$ is the orthogonal projection of $c_1(L) \in H^2(M, \mathbb{R}) = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$ into the space $\mathcal{H}_g^+$ of harmonic self-dual 2-forms, defined with respect to g. Since ω is assumed to be self-dual with respect to g, we therefore have

$$[c_1(L)^+]^2 \geq \frac{(c_1 \cdot [\omega])^2}{[\omega]^2}$$

by the Cauchy-Schwarz inequality. Inequality (3) follows. Since yet another Cauchy-Schwarz argument shows that

$$\int \left(\frac{s^2}{24} + \frac{1}{2}|W_+|^2\right) d\mu_g \geq \frac{1}{36} \int \left(s - \sqrt{6}|W_+|\right)^2 d\mu_g$$

the second inequality and (7) together imply (4). The fact that only Kähler metrics can saturate (3) or (4) is then deduced by examining the relevant Weitzenböck formulæ.

One might be tempted to expect the story to be similar for the norm of the full Riemann tensor. After all, the identity

$$\int_M |\mathcal{R}|^2 d\mu = -8\pi^2(\chi + 3\tau)(M) + 2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

certainly provides a good analog of (7). In the Kähler case, one has

$$\frac{s^2}{24} = |W_+|^2,$$

so this simplifies to become

$$\int_M |\mathcal{R}|^2 d\mu = 8\pi^2(c_2 - c_1^2) + \frac{1}{4}\int_M s^2 \, d\mu_g \ ,$$

and applying (1) we therefore obtain Calabi's inequality

$$\int_M |\mathcal{R}|^2 d\mu \geq 8\pi^2 \left[\frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + c_2 - c_1^2\right] \tag{8}$$

for any Kähler metric. In light of Theorem 1, it might therefore seem reasonable to hope that one could simply extend this inequality to general Riemannian metrics by means of Seiberg-Witten theory. However, we will now show that this cannot work. The key idea is to examine certain extremal Kähler metrics from the vantage point of their *reversed orientations.*

Theorem 2 *Calabi's inequality (8) cannot possibly be extended to general Riemannian metrics by means of Seiberg-Witten theory. Indeed, there actually exist smooth compact oriented Riemannian 4-manifolds (M, g) which admit a spin^c structure of almost-complex type with non-zero Seiberg-Witten invariant, but such that*

$$\int_M |\mathcal{R}|^2 d\mu < 8\pi^2 \left[\frac{(c_1 \cdot [\omega])^2}{[\omega]^2} - (\chi + 3\tau)(M)\right]$$

for some self-dual harmonic 2-form ω on (M, g).

Proof. A *Kodaira-fibered complex surface* is by definition a compact complex surface X equipped with a holomorphic submersion $\varpi : X \to \mathcal{B}$ onto a compact complex curve, such that the base $\mathcal{B}$ and fiber $\mathcal{F}_z = \varpi^{-1}(z)$ both have genus ≥ 2. The product $\mathcal{B} \times \mathcal{F}$ of two complex curves of genus ≥ 2 is certainly Kodaira fibered, but such a product signature $\tau = 0$. However, one can also construct examples (Atiyah 1969, Kodaira 1967) with $\tau > 0$ by taking *branched covers* of products.

Let X be any such Kodaira-fibered surface with $\tau(X) > 0$, and let $\varpi : X \to \mathcal{B}$ be its Kodaira fibration. Let p denote the the genus of the base $\mathcal{B}$, and let q denote the genus of some fiber $\mathcal{F}$ of ϖ. A beautiful result of Fine (2004) then asserts that X actually admits a family of extremal Kähler metrics; namely, *for any sufficiently small $\epsilon > 0$,*

$$[\omega_\epsilon] = 2(p-1)\mathcal{F} - \epsilon c_1$$

is a Kähler class on X which is represented by a Kähler metric g_ϵ of constant scalar curvature.

These metrics, being Kähler, have total scalar curvature

$$\int s_{g_\epsilon} d\mu_{g_\epsilon} = 4\pi c_1 \cdot [\omega_\epsilon] = -4\pi(\chi + \epsilon c_1^2)(X)$$

and total volume

$$\int d\mu_{g_\epsilon} = \frac{[\omega_\epsilon]^2}{2} = \frac{\epsilon}{2}(2\chi + \epsilon c_1^2)(X).$$

Since s_{g_ϵ} is constant, it follows that

$$\int s_{g_\epsilon}^2 d\mu_{g_\epsilon} = \frac{32\pi^2}{\epsilon} \frac{(\chi + \epsilon c_1^2)^2}{2\chi + \epsilon c_1^2}.$$

These metrics therefore satisfy

$$\begin{aligned}\int_X |\mathcal{R}|^2_{g_\epsilon} d\mu_{g_\epsilon} &= 8\pi^2(c_2 - c_1^2) + \frac{1}{4}\int_X s^2 \, d\mu_g \\ &= 8\pi^2\left[-(\chi + 3\tau)(X) + \frac{(\chi + \epsilon c_1^2)^2}{\epsilon(2\chi + \epsilon c_1^2)}\right]\end{aligned}$$

On the other hand, there are symplectic forms on X which are compatible with the *non-standard* orientation of X; for example, the cohomology class $\mathcal{F} + \varepsilon c_1$ is represented by such forms if ε is sufficiently small. A celebrated theorem of Taubes (1994) therefore tells us that the reverse-oriented version $M = \overline{X}$ of X has a non-trivial Seiberg-Witten invariant (Leung 1996, Kotschick 1998). The relevant spin^c structure on $\overline{X}$ is of almost-complex type, and its first Chern class, which we will denote by $\bar{c}_1$, is given by

$$\bar{c}_1 = c_1 + 4(p-1)\mathcal{F}\,.$$

Since this a $(1,1)$-class, one has

$$(\bar{c}_1)^+ = \frac{\bar{c}_1 \cdot [\omega_\epsilon]}{[\omega_\epsilon]^2}\omega_\epsilon = -\frac{(\chi + 3\epsilon\tau)}{[\omega_\epsilon]^2}\omega_\epsilon,$$

relative to the Kähler metric g_ϵ, so that

$$|(\bar{c}_1)^+|^2 = \frac{(\chi + 3\epsilon\tau)^2}{[\omega_\epsilon]^2} = \frac{(\chi + 3\epsilon\tau)^2}{\epsilon(2\chi + \epsilon c_1^2)}.$$

Now since $\bar{c}_1$ arises from an almost-complex structure on $\overline{X}$, we have

$$|(\bar{c}_1)^-|^2 - |(\bar{c}_1)^+|^2 = 2\chi - 3\tau,$$

so that

$$|(\bar{c}_1)^-|^2 = 2\chi - 3\tau + \frac{(\chi + 3\epsilon\tau)^2}{\epsilon(2\chi + \epsilon c_1^2)},$$

and

$$|(\bar{c}_1)^-|^2 - (\chi - 3\tau)(X) = \chi(X) + \frac{(\chi + 3\epsilon\tau)^2}{\epsilon(2\chi + \epsilon c_1^2)},$$

where, for example, τ indicates $\tau(X)$. But it therefore follows that

$$\frac{1}{8\pi^2}\int_X |\mathcal{R}|^2_{g_\epsilon} d\mu_{g_\epsilon} - \left[|(\bar{c}_1)^-|^2 - (\chi - 3\tau)\right] = -(2\chi + 3\tau) + \frac{(\chi + \epsilon c_1^2)^2 - (\chi + 3\epsilon\tau)^2}{\epsilon(2\chi + \epsilon c_1^2)}$$

$$
\begin{aligned}
&= -(2\chi+3\tau)+\frac{4\chi\epsilon(\chi+3\epsilon\tau)+4\chi^2\epsilon^2}{\epsilon(2\chi+\epsilon c_1^2)}\\
&= -(2\chi+3\tau)+2\chi\frac{2\chi+\epsilon c_1^2+3\epsilon\tau}{2\chi+\epsilon c_1^2}\\
&= -3\tau(X)\left[1-\frac{2\chi\epsilon}{2\chi+\epsilon c_1^2}\right] ,
\end{aligned}
$$

which is negative for any sufficiently small ϵ. The result therefore follows once we take "ω" to be the anti-self-dual harmonic form $(\bar{c}_1)^-$, which becomes self-dual on $M=\overline{X}$. ∎

Similarly, careful examination of these examples also shows that, for any constant $t>1$, the Seiberg-Witten equations cannot imply an estimate of

$$\int \left(s-t\sqrt{6}|W_+|\right)^2 d\mu$$

which is saturated by constant-scalar-curvature Kähler metrics. Of course, the Seiberg-Witten equations still imply lower bounds for such quantities, but they are simply never as sharp as those obtained for $t\in[0,1]$.

In this article, we have seen that constant-scalar-curvature Kähler metrics occupy a privileged position in 4-dimensional Riemannian geometry. I would therefore like to conclude this discussion by indicating a bit of what we now know concerning their existence.

There are several ways to phrase the problem. From the Riemannian point of view, one might want to fix a smooth compact oriented 4-manifold M, and simply ask whether there exists an extremal Kähler metric g, where the associated complex structure is not specified as part of the problem. Since M must in particular admit a Kähler metric, two necessary conditions are that M must admit a complex structure and have even first Betti number. Provided these desiderata are fulfilled, Shu (2006) has then shown that an extremal metric g always exists. For all but two diffeotypes, moreover, one can actually arrange for the extremal Kähler metric g to have constant scalar curvature. However, these two exceptional diffeotypes are $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$, and $\mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$, and it is now known (Chen, LeBrun & Weber 2008) that both these manifolds carry Einstein metrics — indeed, even Einstein metrics which are conformal rescalings of extremal Kähler metrics! Since, in conjunction with $F=0$, any Einstein metric of course satisfies the Einstein-Maxwell equations, we thus immediately deduce the following:

Theorem 3 *Let M be the underlying 4-manifold of any compact complex surface of Kähler type. Then M admits a Riemannian solution (g, F) of the Einstein-Maxwell equations.*

While the above formulation of Shu's result certainly suffices to imply Theorem 3, it unfortunately also obfuscates the nature of the proof, which involves constructing extremal Kähler metrics compatible with some *fixed* complex structure in each possible deformation class. The key tool used for this purpose is due to Arezzo and Pacard (2006, 2008), who have shown that constant-scalar-curvature Kähler metrics can be constructed on blow-ups and desingularizations of constant-scalar-curvature Kähler orbifolds, under only very mild assumptions on the complex automorphism group; similar results moreover have even been proved concerning the strictly extremal case (Arezzo, Pacard & Singer 2007). These gluing results represents a vast generalization of earlier work by the present author and his collaborators (Kim, LeBrun & Pontecorvo 1997, LeBrun 1991, LeBrun & Singer 1993) regarding the limited realm of scalar-flat Kähler surfaces. In fact, by invoking the theory of Kähler-Einstein metrics (Aubin 1976, Yau 1977), Arezzo & Pacard (2006) had already shown that every Kähler-type complex surface of Kodaira dimension 0 or 2 admits compatible constant-scalar-curvature Kähler metrics. Shu's results concerning the remaining cases of Kodaira dimensions $-\infty$ and 1 are much less robust, but still easily produce enough examples to imply most of Theorem 3.

Of course, one ultimately doesn't want to settle for mere existence statements; we would really like to completely understand the moduli space of solutions! From this point of view, the first question to ask is whether there can only be one solution for any given complex structure and Kähler class. Modulo complex automorphisms, uniqueness always holds in this setting, as was proved in a series of a fundamental papers by Donaldson (2001), Mabuchi (2004), and Chen & Tian (2005). For a fixed complex structure, one also knows that the Kähler classes of extremal Kähler metrics sweep out an open subset of the Kähler cone (LeBrun & Simanca 1993), and somewhat weaker results are also available regarding deformations of complex structure (Fujiki & Schumacher 1990, LeBrun & Simanca 1994). However, it turns out that the set of Kähler classes which are representable by extremal Kähler metrics may sometimes be a *proper* non-empty open subset of the Kähler cone (Apostolov & Tønnesen-Friedman 2006, Ross 2006). The latter phenomenon is related to algebro-geometric stability problems (Mabuchi 2005, Ross &

Thomas 2006) in a manner which is still only partly understood, but there is reason to hope that a definitive understanding of such issues may result from the incredible ferment of research currently being carried out the field.

As we saw in Theorem 3, Kähler geometry supplies a natural and beautiful way of constructing solutions of the Einstein-Maxwell equations on many compact 4-manifolds. In the opposite direction, we also have the following easy but rather suggestive result:

Proposition 3 *Let M be the underlying 4-manifold of any compact complex surface of* non-Kähler *type with vanishing geometric genus. Then M does not carry any Riemannian solution of the Einstein-Maxwell equations.*

Proof. Let us begin by remembering the remarkable fact (Siu 1983, Barth, Peters & Van de Ven 1984, Buchdahl 1999) that a compact complex surface is of Kähler type iff it has b_1 even. Consequently, for any non-Kähler-type complex surface M, b_1 is odd, and $b_+ = 2p_{\mathbf{g}}$, where $p_{\mathbf{g}} = h^{2,0}$ is the geometric genus (Barth et al. 1984, Theorem IV.2.6). Since the latter is assumed to vanish, M then has negative-definite intersection form, and Hodge theory tells us that there are no non-trivial self-dual harmonic 2-forms for any metric g on M.

Now suppose that (g, F) is a Riemannian solution of the Einstein-Maxwell equations on M. Then the harmonic 2-form F satisfies $F^+ = 0$, and hence

$$\mathring{r} = -[F \circ F]_0 = -2F^+ \circ F^- = 0.$$

The metric g must therefore be Einstein. But the negative-definiteness of the intersection form also tells us that $(2\chi + 3\tau)(M) = c_1^2(M) \leq 0$. The Hitchin-Thorpe inequality for Einstein manifolds (Hitchin 1974*b*) therefore guarantees that M has a finite normal cover $\tilde{M}$ which is diffeomorphic to either $K3$ or T^4, and so, in particular, has b_1 even. Pulling back the complex structure J of M to this cover, we therefore obtain a complex surface $(\tilde{M}, \tilde{J})$ of Kähler type. Averaging an arbitrary Kähler metric h on $(\tilde{M}, \tilde{J})$ over the finite group of deck transformation of $\tilde{M} \to M$ then gives us a Kähler metric which descends to (M, J). Thus (M, J) is of Kähler type, in contradiction to our hypotheses. Our supposition was therefore false, and M thus cannot carry any Riemannian solution of the Einstein-Maxwell equations. ∎

This article has endeavored to convince the reader that the four-dimensional Einstein-Maxwell equations represent a beautiful and natural

generalization of the constant-scalar-curvature Kähler condition. However, it still remains to be seen whether solutions exist on many compact 4-manifolds other than complex surfaces of Kähler type. In this direction, my guess is that Proposition 3 will actually prove to be rather misleading. For example, there are ALE Riemannian solutions of the Einstein-Maxwell equations, constructed (Yuille 1987) via the Israel-Wilson ansatz, on manifolds very unlike any complex surface. It thus seems reasonable to conjecture that there are plenty of of compact solutions that in no sense arise from Kähler geometry. Perhaps some interested reader will feel inspired to go out and find some!

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